Electromagnetic Instability of a Rotating Electron Layer in a Sheath Helix

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Abstract —A sheath helix supports slow electromagnetic modes with phase velocity considerably lower than the velocity of light in a vacuum. In the presence of a rotating electron layer, the modes can be resonantly driven unstable via cyclotron maser interaction. Using the perturbation technique, the growth rate of the instability is obtained in the weak-beam approximation and is seen to decrease with the slowing down of the modes. For lower order modes, the growth rate is comparable to the one with a cocentric cylindrical waveguide. However, for higher order modes, the growth rate decreases rapidly, suggesting that a sheath helix may be used to suppress the higher order modes.

I. Introduction

Gyrotron and rotating electron-layer magnetrons have recently come up as potential high-power sources of millimeter and submillimeter waves [1]–[5]. The basic mechanism of energy transfer in these devices is the cyclotron maser instability. In the presence of a signal of frequency near any harmonic of electron cyclotron frequency, the electrons tend to bunch in their gyrophases due to the relativistic dependence of mass on velocity. When the bunching is favorable, it tends to enhance the signal, leading to the growth of the instability. The cyclotron resonance is sensitive to the parallel velocity of electrons and also to the parallel wave number of the waveguide mode. Earlier experiments on these devices have employed cylindrical waveguides for cyclotron maser interaction in which the phase velocities of the modes are greater than the velocity of light in a vacuum, and parallel motions do not have an important role. However, it is worthwhile to examine the effect of the slowing down of these modes on cyclotron maser interaction.

Sheath helix is one of the various slow wave structures that have been widely employed in the conventional traveling wave tubes [6]–[8]. Recently, Choe and Uhm [9] have studied the effect of slowing the modes on cyclotron maser interaction by mounting a sheath helix inside a gyrotron. In this device, the guiding centers of the electrons lie on a circle of finite radius. In this paper we investigate the cyclotron maser instability of a rotating electron layer in a sheath helix. The model considered here is different from that of Choe and Uhm in that the guiding centers of the electrons of the rotating layer lie on the axis of the sheath helix. Destler *et al.*, [5] have studied the effect of a slow wave structure for waves traveling transverse to the magnetic field. It has led to the narrowing of the spectrum. Here we consider the effect of the slowing down of the modes in the direction of ambient magnetic field. First, a dispersion relation for the electromagnetic modes in a sheath helix is obtained in cylindrical geometry in the weak-beam approximation (i.e., the density of the beam is too low to alter the modes of the helix) in Section II. Then, the growth rate of the modes is obtained using a well-known perturbation technique. The solution of the dispersion relation and the growth rate are obtained for various helix parameters and are discussed in Section III.

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II. Dispersion Relation

We consider a sheath helix of radius a and pitch L (Cf Fig. 1) having infinite conductivity in the direction of the helix wire i.e., at an angle $\psi = \cot^{-1}(2\pi a/L)$ with the plane normal to the axis of the helix, and zero conductivity in the transverse direction. A thin hollow rotating electron layer of radius r_b propagates along the axis of the system with axial drift velocity V_b and azimuthal velocity $V_{\theta 0}$. The system is immersed in a uniform axial magnetic field B_0.

The sheath helix, due to its anisotropic conductivity, supports mixed TE and TM modes [8]. The fields may be written as

$$\vec{E} = \vec{E}(r)\exp[-i(\omega t - \beta z - m\theta)]$$

$$\vec{B} = \vec{B}(r)\exp[-i(\omega t - \beta z - m\theta)]$$

where ω, β are the angular frequency and parallel wave number of the wave, respectively, and m is the azimuthal mode number. The wave equation governing the propagation of waves in cylindrical geometry can be written as

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\omega^2}{c^2} - \beta^2\right)E_z = -i\mu_0\omega J_z + \frac{i\beta}{\epsilon_0}\rho \quad (1)$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\omega^2}{c^2} - \beta^2\right)B_z = -\mu_0\left(\frac{1}{r}\frac{\partial}{\partial r}rJ_\theta - \frac{imJ_r}{r}\right) \quad (2)$$

where ρ and $\vec{J}$ are charge and current densities. In the weak-beam limit the contribution of the beam on the mode

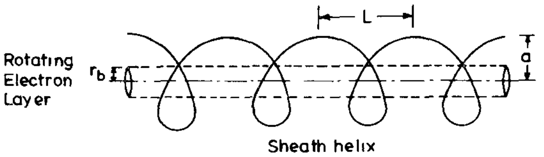

Fig. 1. Schematic of a sheath helix with a rotating electron layer.

structure can be neglected

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}+\frac{\omega^2}{c^2}-\beta^2\right)E_z=0 \tag{3}$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}+\frac{\omega^2}{c^2}-\beta^2\right)B_z=0. \tag{4}$$

The solutions of (3) and (4) inside and outside the helix can be expressed, respectively, as

$$E_{zm}^{(i)}=A_m I_m(\xi_m r)\exp[-i(\omega t-\beta z-m\theta)] \tag{5}$$

$$B_{zm}^{(i)}=C_m I_m(\xi_m r)\exp[-i(\omega t-\beta z-m\theta)] \tag{6}$$

$$E_{zm}^{(0)}=B_m K_m(\xi_m r)\exp[-i(\omega t-\beta z-m\theta)] \tag{7}$$

$$B_{zm}^{(0)}=D_m K_m(\xi_m r)\exp[-i(\omega t-\beta z-m\theta)] \tag{8}$$

where $\xi_m^2=\beta^2-k^2$; $k(=\omega/c)$ is the free-space wave number; I_m, K_m are the modified Bessel functions of order m. The superscripts (i) and (0) refer to the inside and outside regions. From (5)–(8), on using Maxwell's equations, the r and θ components of the electric and magnetic fields in the inside and outside regions can be determined.

Applying boundary conditions on the field components appropriate to the case of sheath helix [8] and eliminating the constants A_m, B_m, C_m, D_m, a dispersion relation describing the relationship between ω and β of the modes of the sheath helix can be written as

$$\beta^2\left[m+\frac{a\xi_m^2}{\beta}\tan\psi\right]^2 I_m(\xi_m a)K_m(\xi_m a)$$
$$=k^2[m\xi_m+a\xi_m I_{m+1}(\xi_m a)]$$
$$\times[a\xi_m K_{m+1}(\xi_m a)-mK_m(\xi_m a)]. \tag{9}$$

Now we study the interaction of the rotating electron layer with modes of the helix. The equation of motion and the equation of continuity for electrons can be written as

$$m\frac{d(\gamma\vec{v})}{dt}=-e\left[\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right] \tag{10}$$

$$\frac{\partial n}{\partial t}+\vec{\nabla}\cdot(n\vec{v})=0 \tag{11}$$

where $m, -e$ are the mass and charge of the beam electrons; $\gamma[=(1-\beta_r^2-\beta_\theta^2-\beta_z^2)^{-1/2}]$ is the relativistic mass ratio with $\beta_r=v_r/c$, $\beta_\theta=v_\theta/c$, $\beta_z=v_z/c$. The velocity components in the equilibrium state obtained from (10) are

$$v_{r0}=0$$
$$v_{\theta 0}=\left(\frac{eB_0}{m\gamma_0}\right)r_b=\omega_c r_b$$
$$v_{z0}=V_b \tag{12}$$

where

$$\gamma_0=\left(1-\beta_{\theta 0}^2-\beta_{z0}^2\right)^{-1/2}.$$

In the presence of the electromagnetic eigenmodes fields, the particle positions and velocities can be expressed as

$$r=r_0+r_1(\theta,z,t) \tag{13a}$$

$$\theta=\theta_0+\omega_c t+\theta_1(\theta,z,t) \tag{13b}$$

$$z=z_0+V_b t+z_1(\theta,z,t) \tag{13c}$$

$$v_{r1}=\dot{r}_1 \tag{14a}$$

$$v_{\theta 1}=r_0\dot{\theta}_1+\omega_c r_1 \tag{14b}$$

$$v_{z1}=\dot{z}_1. \tag{14c}$$

Expressing space-time variations to be of the form $\sim\exp[-i(\omega t-\beta z-m\theta)]$, the perturbed quantities can be written as

$$r_1=\frac{e}{m\gamma_0\alpha_m^2}\left[\phi_m^2 E_r+i\phi_m\omega_c E_\theta+i\phi_m\omega_c\beta_{z0}cB_r\right.$$
$$\left.-\phi_m^2\beta_{z0}cB_\theta+\phi_m^2\beta_{\theta 0}cB_z\right] \tag{15a}$$

$$r_0\theta_1=\frac{ie}{m\gamma_0\alpha_m^2}\left[-\phi_m\omega_c E_r-i\left(\frac{\phi_m^2}{\gamma_{\theta 0}^2}+\omega_c^2\beta_{\theta 0}^2\right)E_\theta\right.$$
$$+i\beta_{\theta 0}\beta_{z0}\left(\phi_m^2-\omega_c^2\right)E_z-i\beta_{z0}c\phi_m^2 B_r$$
$$\left.+\phi_m\omega_c\beta_{z0}cB_\theta-\phi_m\omega_c\beta_{\theta 0}cB_z\right] \tag{15b}$$

$$z_1=\frac{e}{m\gamma_0\phi_m^2}\left[\frac{1}{\gamma_{z0}^2}E_z-\beta_{z0}\beta_{\theta 0}E_\theta-\beta_{\theta 0}cB_r\right] \tag{15c}$$

where

$$\alpha_m=\phi_m^2\left(\phi_m^2-\omega_c^2\right)$$
$$\phi_m=\omega-\beta V_b-m\omega_c$$
$$\gamma_{z0}=\left(1-\beta_{z0}\right)^{-1/2}$$
$$\gamma_{\theta 0}=\left(1-\beta_{\theta 0}\right)^{-1/2}.$$

The linearized velocity components become

$$v_{r1}=-i\phi_m r_1$$
$$v_{\theta 1}=-i\phi_m r\theta_1+\omega_c r_1.$$
$$v_{z1}=-i\phi_m z_1. \tag{16}$$

We assume the beam to be thin and its density of the form $n=n_b\delta(r-r_b)$ where n_b is surface density of the beam electrons. The linearized charge and current densities obtained with the help of (11) can be written as [4]

$$\rho=f\delta(r-r_b)+en_b\delta'(r-r_b)r_1$$
$$\vec{J}=-en_b\delta(r-r_b)\vec{v}_1+f\delta(r-r_b)\vec{v}_0$$
$$+en_b\delta'(r-r_b)r_1\vec{v}_0 \tag{17}$$

where

$$f=en_b\left[\frac{(\phi_m-m\omega_c)}{\phi_m}\frac{r_1}{r_b}+im\theta_1+i\beta z_1\right]. \tag{18}$$

Substituting ρ and $\vec{J}$ from (17) in (1), expressing the axial component of the field as $E_z = \Sigma_m E_{zm}$, where E_{zm} is given by (5), multiplying the resulting equation by $E_{zm}^* r\,dr$ and integrating from 0 to $+\infty$, we obtain a dispersion relation which in the limit $\phi_m \to 0$ reduces to

$$(\xi^2 - \xi_m^2)\phi_m^2 = XA_m^2[yF_1 + F_2 + F_3]I_m^2(\xi_m r_b) + XA_m^2 YF_4 I_m(\xi_m r_b) I_{m+1}(\xi_m r_b) \quad (19)$$

where

$$X = \frac{\omega_b^2\left(\beta - \dfrac{\omega V_b}{c^2}\right)}{r_b^2 \gamma_0 \xi_m^2}$$

$$y = \frac{\left(m + \dfrac{a\xi_m^2 \tan\psi}{\beta}\right)}{\left(m + a\xi_m \dfrac{I_{m+1}(\xi_m a)}{I_m(\xi_m a)}\right)}$$

$$\omega_b^2 = \frac{4\pi n_b e^2}{m r_b}$$

$$F_1 = m^2\left(\beta - \frac{\beta^2 V_b}{\omega}\right) - m^2\beta_{\theta 0}^2\beta - m r_b \beta^2 \beta_{z0}\beta_{\theta 0} + m\beta_{\theta 0} c r_b \frac{\partial^3}{\omega}$$

$$F_2 = -m^2\left(\beta - \frac{\omega V_b}{c^2}\right) + m^2\beta\beta_{\theta 0}^2 + m\beta_{\theta 0}\beta_{z0} r_b \beta^2 - m\beta_{\theta 0}\beta \frac{\omega}{c} r_b$$

$$F_3 = -\beta \frac{\xi_m^2 r_b^2}{\gamma_{z0}^2} + m\beta_{\theta 0}\beta_{z0}\xi_m^2 r_b$$

$$F_4 = m\beta\xi_m\left(1 - \frac{\beta V_b}{\omega}\right) r_b - m\xi_m \beta_{\theta 0}^2 r_b \beta - \beta^2 \xi_m \beta_{z0}\beta_{\theta 0} r_b^2 + \beta^3 \frac{\xi_m \beta_{\theta 0} c}{\omega} r_b^2$$

and A_m is determined by the normalization condition

$$\int_0^\infty E_{zm} E_{zm}^* r\,dr = 1.$$

The left-hand side of (19) contains two factors. In the limit of vanishing beam density, the first factor equated to zero gives the mode

$$\xi^2 - \xi_m^2 = 0 \quad (20)$$

of the helix and the second factor equal to zero gives the beam mode

$$\omega - \beta V_b - m\omega_c = 0. \quad (21)$$

These two modes are coupled via the term on the right-hand side in (19). The instability occurs when the two factors are simultaneously zero i.e., the dispersion (ω, β) curves of the two modes intersect each other. Let the value of ω at the interaction point be ω_r, which is a solution of (20) and (21). Assuming $\omega(=\omega_r + i\omega_i)$ to be complex with $\omega_i \ll \omega_r$ and expanding ξ as

$$\xi = \xi_m + \frac{\partial \xi}{\partial \omega}(\omega - \omega_m). \quad (22)$$

Equation (19) gives the following expression for the growth rate

$$\omega_i = \left[\frac{XA_m^2 c}{2\omega_r}\{I_m^2(\xi_m r_b)\cdot(YF_1 + F_2 + F_3) + I_m(\xi_m r_b) I_{m+1}(\xi_m r_b) YF_4\}\right]^{1/3} \cdot \cos\frac{(4n+1)\pi}{6} \quad (23)$$

where $n = 0, 1, 2, \cdots$.

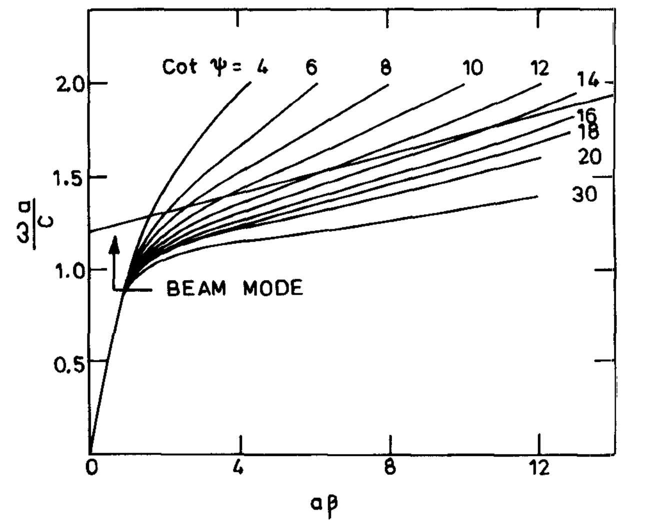

Fig. 2. Dispersion characteristics of the sheath helix for $m = 1$ mode with $\cot\psi$ as a parameter. The solid line represents a beam mode for $V_b/c = 0.05$, $a\omega_c/c = 1.2$.

III. Results and Discussion

Fig. 2 shows the dispersion characteristics of the sheath helix for $m = 1$ mode, obtained by solving (9) with $\cot\psi$ as a parameter. It can be seen that with increasing $\cot\psi$, the phase velocity of the mode decreases. Also shown in Fig. 2 is the beam mode obtained from (21) for $V_b/c = 0.05$, $a\omega_c/c = 1.2$ and $m = 1$. The resonant interaction of the two modes results in the instability. Determining ω, β from the intersection points, the growth rate of the instability evaluated from (23) is plotted as a function of $\cot\psi$ in Fig. 3, only for those modes whose wavelengths are much greater than the pitch of the helix i.e., $[a\beta/\cot\psi] \ll 1$. The rate of instability decreases with the slowing down of the modes.

Calculations were made for several m values and it was found that the growth rate for lower order modes is comparable with that obtained for a cocentric cylindrical waveguide [4]. However, for higher order modes the growth

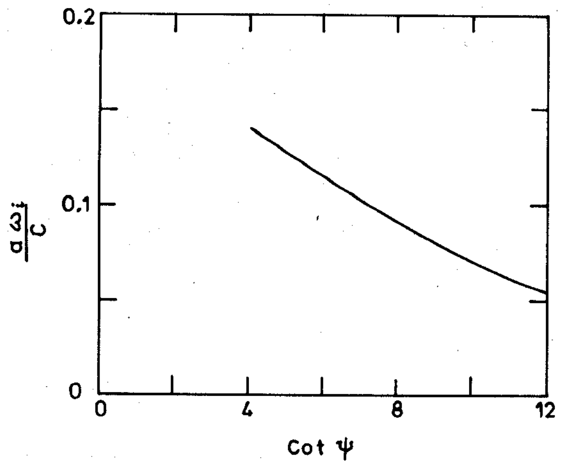

Fig. 3. Variation of the growth rate with cot ψ. The helix and beam parameters are $a = 7.5$ cm, $\omega_b/\omega_c = 0.1$, $r_b = 6$ cm.

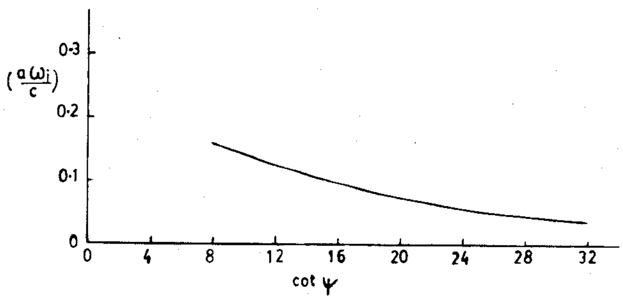

Fig. 4. Variation of the growth rate with cot ψ for backward interaction for $V_b/c = -0.2$, $a = 15$ cm. Other parameters are the same as in Fig. 3.

rate decreases rapidly. For example, the growth rate $(a\omega_i/c)$ of $m = 8$ mode is 0.0076, 0.0065 for cot $\psi = 30$ and 40, respectively. It should be noted here that for $m = 8$, the resonant interaction between the beam and helix modes is possible only when the radius of the helix is reduced to $a = 6.4$ cm. This implies that the higher order modes can be suppressed by suitably choosing the radius of the helix. Calculations were also made for backward interaction by choosing the beam to be propagating in the opposite direction of the wave. Fig. 4 shows the variation of the growth rate with cot ψ for $V_b/c = -0.2$, $\omega_b/\omega_c = 0.1$, $r_b = 6$ cms, $a = 15$ cms, $\gamma = 6$. It is seen that the growth rate decreases with slowing of the modes for the case of backward interaction also. In the analysis presented here, the effect of metallic walls on the mode structure and growth rate of the instability has not been included. However, if the walls are at a distance considerably larger than the helix radius, the effect due to walls can be neglected as the mode amplitude decays sharply off the helix boundary.

To summarize, we have studied the cyclotron maser instability of a relativistic electron layer in a sheath helix. This study was prompted by recent investigations of Destler *et al.*, who employed the fast wave coupling of the cocentric waveguide mode with the cyclotron frequency upshifted beam mode (cyclotron maser interaction) to generate high-power microwaves. With a view to investigate the effect of slowing the wave modes on the cyclotron maser interaction, a sheath helix has been considered in place of a cocentric waveguide in our model. The results of our analysis show that the growth rate of the instability decreases with a slowing of the modes, implying that the sheath helix with a rotating electron layer is less efficient as a microwave source.

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